

Mathematical Background

Outline

Sets

Relations

Functions

Products

Sums

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Products

Sums

Sets – Basic Notations

$x \in S$	membership
$S \subseteq T$	subset
$S \subset T$	proper subset
$S \subseteq^{\text{fin}} T$	finite subset
$S = T$	equivalence
$\emptyset$	the empty set
N	natural numbers
Z	integers
B	{true, false}

Sets – Basic Notations

$S \cap T$	intersection	$\stackrel{\text{def}}{=} \{x \mid x \in S \text{ and } x \in T\}$
$S \cup T$	union	$\stackrel{\text{def}}{=} \{x \mid x \in S \text{ or } x \in T\}$
$S - T$	difference	$\stackrel{\text{def}}{=} \{x \mid x \in S \text{ and } x \notin T\}$
$\mathcal{P}(S)$	powerset	$\stackrel{\text{def}}{=} \{T \mid T \subseteq S\}$
$[m, n]$	integer range	$\stackrel{\text{def}}{=} \{x \mid m \leq x \leq n\}$

Generalized Unions of Sets

$$\bigcup \mathcal{S} \stackrel{\text{def}}{=} \{x \mid \exists T \in \mathcal{S}. x \in T\}$$

$$\bigcup_{i \in I} S(i) \stackrel{\text{def}}{=} \bigcup \{S(i) \mid i \in I\}$$

$$\bigcup_{i=m}^n S(i) \stackrel{\text{def}}{=} \bigcup_{i \in [m,n]} S(i)$$

Here $\mathcal{S}$ is a set of sets. $S(i)$ is a set whose definition depends on i . For instance, we may have

$$S(i) = \{x \mid x > i + 3\}$$

Given $i = 1, 2, \dots, n$, we know the corresponding $S(i)$.

Generalized Unions of Sets

Example (1)

$$A \cup B = \bigcup \{A, B\}$$

Proof?

Example (2)

Let $S(i) = [i, i + 1]$ and $I = \{j^2 \mid j \in [1, 3]\}$, then

$$\bigcup_{i \in I} S(i) = \{1, 2, 4, 5, 9, 10\}$$

Generalized Intersections of Sets

$$\bigcap \mathcal{S} \stackrel{\text{def}}{=} \{x \mid \forall T \in \mathcal{S}. x \in T\}$$

$$\bigcap_{i \in I} S(i) \stackrel{\text{def}}{=} \bigcap \{S(i) \mid i \in I\}$$

$$\bigcap_{i=m}^n S(i) \stackrel{\text{def}}{=} \bigcap_{i \in [m,n]} S(i)$$

Generalized Unions and Intersections of Empty Sets

From

$$\bigcup \mathcal{S} \stackrel{\text{def}}{=} \{x \mid \exists T \in \mathcal{S}. x \in T\}$$

$$\bigcap \mathcal{S} \stackrel{\text{def}}{=} \{x \mid \forall T \in \mathcal{S}. x \in T\}$$

we know

$$\bigcup \emptyset = \emptyset$$

$$\bigcap \emptyset \quad \text{meaningless}$$

$\bigcap \emptyset$ is meaningless, since it denotes the paradoxical “set of everything” (see Russell’s paradox).

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Relations

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Products

Sums

Relations

We need to first define the *Cartesian product* of two sets A and B :

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

Here (x, y) is called a *pair*.

Projections over pairs:

$$\pi_0(x, y) = x \text{ and } \pi_1(x, y) = y.$$

Then, ρ is a *relation from A to B* if $\rho \subseteq A \times B$.

Or, written as $\rho \in \mathcal{P}(A \times B)$.

Relations

ρ is a *relation from A to B* if $\rho \subseteq A \times B$, or $\rho \in \mathcal{P}(A \times B)$.

ρ is a *relation on S* if $\rho \subseteq S \times S$.

We say ρ *relates x and y* if $(x, y) \in \rho$. Sometimes we write it as $x \rho y$.

ρ is an *identity relation* if $\forall (x, y) \in \rho. x = y$.

Relations – Basic Notations

the *identity* on S $\text{Id}_S \stackrel{\text{def}}{=} \{(x, x) \mid x \in S\}$

the *domain* of ρ $\text{dom}(\rho) \stackrel{\text{def}}{=} \{x \mid \exists y. (x, y) \in \rho\}$

the *range* of ρ $\text{ran}(\rho) \stackrel{\text{def}}{=} \{y \mid \exists x. (x, y) \in \rho\}$

composition of ρ and ρ' $\rho' \circ \rho \stackrel{\text{def}}{=} \{(x, z) \mid \exists y. (x, y) \in \rho \wedge (y, z) \in \rho'\}$

inverse of ρ $\rho^{-1} \stackrel{\text{def}}{=} \{(y, x) \mid (x, y) \in \rho\}$

Relations – Properties and Examples

$$(\rho_3 \circ \rho_2) \circ \rho_1 = \rho_3 \circ (\rho_2 \circ \rho_1)$$

$$\rho \circ \text{Id}_S = \rho = \text{Id}_T \circ \rho, \text{ if } \rho \subseteq S \times T$$

$$\text{dom}(\text{Id}_S) = S = \text{ran}(\text{Id}_S)$$

$$\text{Id}_T \circ \text{Id}_S = \text{Id}_{T \cap S}$$

$$\text{Id}_S^{-1} = \text{Id}_S$$

$$(\rho^{-1})^{-1} = \rho$$

$$(\rho_2 \circ \rho_1)^{-1} = \rho_1^{-1} \circ \rho_2^{-1}$$

$$\rho \circ \emptyset = \emptyset = \emptyset \circ \rho$$

$$\text{Id}_{\emptyset} = \emptyset = \emptyset^{-1}$$

$$\text{dom}(\rho) = \emptyset \iff \rho = \emptyset$$

Relations – Properties and Examples

$$< \subseteq \leq$$

$$< \cup \text{Id}_{\mathbf{N}} = \leq$$

$$\leq \cap \geq = \text{Id}_{\mathbf{N}}$$

$$< \cap \geq = \emptyset$$

$$< \circ \leq = <$$

$$\leq \circ \leq = \leq$$

$$\geq = \leq^{-1}$$

Equivalence Relations

ρ is an *equivalence relation* on S if it is reflexive, symmetric and transitive.

Reflexivity: $\text{Id}_S \subseteq \rho$

Symmetry: $\rho^{-1} = \rho$

Transitivity: $\rho \circ \rho \subseteq \rho$

Outline

Sets

Relations

Functions

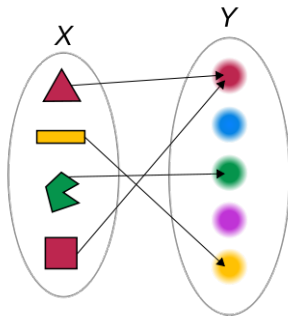
Products

Sums

Functions

A function f from A to B is a special relation from A to B .

A relation ρ is a function if, for all x, y and y' , $(x, y) \in \rho$ and $(x, y') \in \rho$ imply $y = y'$.



Function application $f(x)$ can also be written as $f x$.

Below we only consider *total* functions: $\text{dom}(f) = A$.

Functions

$\emptyset$ and Id_S are functions.

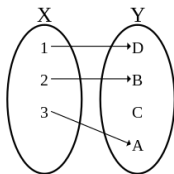
If f and g are functions, then $g \circ f$ is a function.

$$(g \circ f)x = g(fx)$$

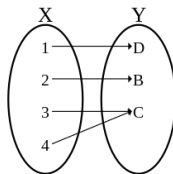
If f is a function, f^{-1} is *not* necessarily a function. (f^{-1} is a function if f is an injection.)

Functions – Injection, Surjection and Bijection

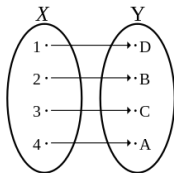
Injective and non-surjective:



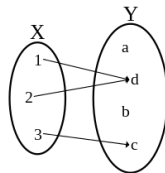
Surjective and non-injective:



Bijjective:



Non-injective and non-surjective:



Functions – Denoted by Typed Lambda Expressions

$\lambda x \in S. E$ denotes the function f with domain S such that $f(x) = E$ for all $x \in S$.

Example

$\lambda x \in \mathbf{N}. x + 3$ denotes the function $\{(x, x + 3) \mid x \in \mathbf{N}\}$.

Functions – Variation

Variation of a function at a single argument:

$$f\{x \rightsquigarrow n\} \stackrel{\text{def}}{=} \lambda z. \begin{cases} f\ z & \text{if } z \neq x \\ n & \text{if } z = x \end{cases}$$

Note that x does not have to be in $\text{dom}(f)$.

$$\text{dom}(f\{x \rightsquigarrow n\}) = \text{dom}(f) \cup \{x\}$$

$$\text{ran}(f\{x \rightsquigarrow n\}) = \text{ran}(f - \{(x, n') \mid (x, n') \in f\}) \cup \{n\}$$

Example

$$(\lambda x \in [0, 2]. x + 1)\{2 \rightsquigarrow 7\} = \{(0, 1), (1, 2), (2, 7)\}$$

$$(\lambda x \in [0, 1]. x + 1)\{2 \rightsquigarrow 7\} = \{(0, 1), (1, 2), (2, 7)\}$$

Function Types

We use $A \rightarrow B$ to represent the set of all functions from A to B .

$\rightarrow$ is right associative. That is,

$$A \rightarrow B \rightarrow C = A \rightarrow (B \rightarrow C).$$

If $f \in A \rightarrow B \rightarrow C$, $a \in A$ and $b \in B$, then $f\ a\ b = (f(a))b \in C$.

Functions with multiple arguments

$$f \in A_1 \times A_2 \times \cdots \times A_n \rightarrow A$$

$$f = \lambda x \in A_1 \times A_2 \times \cdots \times A_n. E$$

$$f(a_1, a_2, \dots, a_n)$$

Currying it gives us a function

$$g \in A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow A$$

$$g = \lambda x_1 \in A_1. \lambda x_2 \in A_2. \dots \lambda x_n \in A_n. E$$

$$g\ a_1\ a_2\ \dots\ a_n$$

Outline

Sets

Relations

Functions

Products

Sums

Cartesian Products

Recall $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

Projections over pairs: $\pi_0(x, y) = x$ and $\pi_1(x, y) = y$.

Generalize to n sets:

$$S_0 \times S_1 \times \cdots \times S_{n-1} = \{(x_0, \dots, x_{n-1}) \mid \forall i \in [0, n-1]. x_i \in S_i\}$$

We say $(x_0, \dots, x_{n-1})$ is an *n-tuple*.

Then we have $\pi_i(x_0, \dots, x_{n-1}) = x_i$.

Tuples as Functions

We can view a pair (x, y) as a function

$$\lambda i \in \mathbf{2}. \begin{cases} x & \text{if } i = 0 \\ y & \text{if } i = 1 \end{cases}$$

where $\mathbf{2} = \{0, 1\}$.

$$A \times B \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = \mathbf{2}, \text{ and } f\ 0 \in A \text{ and } f\ 1 \in B\}$$

(We re-define $A \times B$ in this way, in order to generalize $\times$ later. Functions and relations are still defined based on the old definitions of $\times$.)

Tuples as Functions

Similarly, we can view an n -tuple $(x_0, \dots, x_{n-1})$ as a function

$$\lambda i \in \mathbf{n}. \begin{cases} x_0 & \text{if } i = 0 \\ \dots & \dots \\ x_{n-1} & \text{if } i = n - 1 \end{cases}$$

where $\mathbf{n} = \{0, 1, \dots, n - 1\}$.

$$S_0 \times \dots \times S_{n-1} \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = \mathbf{n}, \text{ and } \forall i \in \mathbf{n}. f\ i \in S_i\}$$

Generalized Products

From

$$S_0 \times \cdots \times S_{n-1} \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = \mathbf{n}, \text{ and } \forall i \in \mathbf{n}. f(i) \in S_i\}$$

we can generalize $S_0 \times \cdots \times S_{n-1}$ to an infinite number of sets.

$$\prod_{i \in I} S(i) \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = I, \text{ and } \forall i \in I. f(i) \in S(i)\}$$

$$\prod_{i=m}^n S(i) \stackrel{\text{def}}{=} \prod_{i \in [m, n]} S(i)$$

Generalized Products

Let θ is a function from a set of indices to a set of sets, i.e., θ is an indexed family of sets. We can define $\prod \theta$ as follows.

$$\prod \theta \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = \text{dom}(\theta), \text{ and } \forall i \in \text{dom}(\theta). f(i) \in \theta(i)\}$$

Example

Let $\theta = \lambda i \in I. S(i)$. Then

$$\prod \theta = \prod_{i \in I} S(i)$$

Generalized Products – Examples

$$\prod \theta \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = \text{dom}(\theta), \text{ and } \forall i \in \text{dom}(\theta). f i \in \theta i\}$$

Example (1)

Let $\theta = \lambda i \in \mathbf{2}. \mathbf{B}$. Then

$$\prod \theta = \{ \{(0, \mathbf{true}), (1, \mathbf{true})\}, \\ \{(0, \mathbf{true}), (1, \mathbf{false})\}, \\ \{(0, \mathbf{false}), (1, \mathbf{true})\}, \\ \{(0, \mathbf{false}), (1, \mathbf{false})\} \}$$

That is, $\prod \theta = \mathbf{B} \times \mathbf{B}$.

(Here $\mathbf{B} \times \mathbf{B}$ uses the new definition of $\times$. If we use its old definition, we will see an elegant correspondence between $\prod \theta$ and $\mathbf{B} \times \mathbf{B}$.)

Generalized Products – Examples

$$\prod \theta \stackrel{\text{def}}{=} \{f \mid \text{dom}(f) = \text{dom}(\theta), \text{ and } \forall i \in \text{dom}(\theta). f\ i \in \theta\ i\}$$

Example (2)

$$\prod \emptyset = \{\emptyset\}.$$

Example (3)

If $\exists i \in \text{dom}(\theta). \theta\ i = \emptyset$, then $\prod \theta = \emptyset$.

Exponentiation

Recall $\prod_{x \in T} S(x) = \prod_{\lambda x \in T} S(x)$.

We write S^T for $\prod_{x \in T} S$ if S is independent of x .

$$\begin{aligned} S^T &= \prod_{x \in T} S = \prod_{\lambda x \in T} S \\ &= \{f \mid \text{dom}(f) = T, \text{ and } \forall x \in T. f(x) \in S\} = (T \rightarrow S) \end{aligned}$$

Recall that $T \rightarrow S$ is the set of all functions from T to S .

Exponentiation – Example

We sometimes use 2^S for powerset $\mathcal{P}(S)$. Why?

Exponentiation – Example

We sometimes use 2^S for powerset $\mathcal{P}(S)$. Why?

$$2^S = (S \rightarrow 2)$$

For any subset T of S , we can define

$$f = \lambda x \in S. \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \in S - T \end{cases}$$

Then $f \in (S \rightarrow 2)$.

On the other hand, for any $f \in (S \rightarrow 2)$, we can construct a subset of S .

Outline

Sets

Relations

Functions

Products

Sums

Sums (or Disjoint Unions)

Example

Let $A = \{1, 2, 3\}$ and $B = \{2, 3\}$.

To define the disjoint union of A and B , we need to index the elements according to which set they originated in:

$$A' = \{(0, 1), (0, 2), (0, 3)\}$$

$$B' = \{(1, 2), (1, 3)\}$$

$$A + B = A' \cup B'$$

Sums (or Disjoint Unions)

$$A + B \stackrel{\text{def}}{=} \{(i, x) \mid i = 0 \text{ and } x \in A, \text{ or } i = 1 \text{ and } x \in B\}$$

Injection operations:

$$\iota_{A+B}^0 \in A \rightarrow A + B$$

$$\iota_{A+B}^1 \in B \rightarrow A + B$$

The sum can be generalized to n sets:

$$S_0 + S_1 + \cdots + S_{n-1} \stackrel{\text{def}}{=} \{(i, x) \mid i \in \mathbf{n} \text{ and } x \in S_i\}$$

Generalized Sums (or Disjoint Unions)

It can also be generalized to an infinite number of sets.

$$\sum_{i \in I} S(i) \stackrel{\text{def}}{=} \{(i, x) \mid i \in I \text{ and } x \in S(i)\}$$
$$\sum_{i=m}^n S(i) \stackrel{\text{def}}{=} \sum_{i \in [m, n]} S(i)$$

The sum of θ is

$$\Sigma \theta \stackrel{\text{def}}{=} \{(i, x) \mid i \in \text{dom}(\theta) \text{ and } x \in \theta i\}$$

So

$$\sum_{i \in I} S(i) = \Sigma \lambda i \in I. S(i)$$

Generalized Sums (or Disjoint Unions) – Examples

$$\Sigma \theta \stackrel{\text{def}}{=} \{(i, x) \mid i \in \text{dom}(\theta) \text{ and } x \in \theta i\}$$

Example (1)

$$\sum_{i \in \mathbf{n}} S(i) = \Sigma \lambda i \in \mathbf{n}. S(i) = \{(i, x) \mid i \in \mathbf{n} \text{ and } x \in S(i)\}$$

Example (2)

Let $\theta = \lambda i \in \mathbf{2}. \mathbf{B}$. Then

$$\Sigma \theta = \{ (0, \mathbf{true}), (0, \mathbf{false}), (1, \mathbf{true}), (1, \mathbf{false}) \}$$

That is, $\Sigma \theta = \mathbf{2} \times \mathbf{B}$.

Generalized Sums (or Disjoint Unions) – Examples

$$\Sigma \theta \stackrel{\text{def}}{=} \{(i, x) \mid i \in \text{dom}(\theta) \text{ and } x \in \theta i\}$$

Example (3)

$$\Sigma \emptyset = \emptyset.$$

Example (4)

If $\forall i \in \text{dom}(\theta). \theta i = \emptyset$, then $\Sigma \theta = \emptyset$.

Example (5)

Let $\theta = \lambda i \in \mathbf{2}. \begin{cases} \mathbf{B} & \text{if } i = 0 \\ \emptyset & \text{if } i = 1 \end{cases}$,
then $\Sigma \theta = \{(0, \mathbf{true}), (0, \mathbf{false})\}$.

More on Generalized Sums (or Disjoint Unions)

$$\Sigma \theta \stackrel{\text{def}}{=} \{(i, x) \mid i \in \text{dom}(\theta) \text{ and } x \in \theta i\}$$

$$\sum_{x \in T} S(x) \stackrel{\text{def}}{=} \Sigma \lambda x \in T. S(x)$$

We can prove $\sum_{x \in T} S = T \times S$ if S is independent of x .

$$\begin{aligned} \sum_{x \in T} S &= \Sigma \lambda x \in T. S \\ &= \{(x, y) \mid x \in T \text{ and } y \in S\} = (T \times S) \end{aligned}$$

Quizzes

Quiz 1: True or false?

Let A and B be two sets. Then

$$(A \rightarrow B) \in \mathcal{P}(A \times B).$$

Quiz 2: True or false?

We define the relation $\bowtie$ between two functions $f, g \in \mathbf{N} \rightarrow \mathbf{N}$ as follows:

$$f \bowtie g \text{ iff } \forall x, y. (f(x) = 42) \wedge (g(y) = 42) \implies (x = y).$$

Then $\bowtie$ is transitive, that is,

$$\forall f, g, h. (f \bowtie g) \wedge (g \bowtie h) \implies (f \bowtie h).$$

Quiz 3: True or false?

Let

$$H = \bigcup_{S \subseteq^{\text{fin}} \mathbf{N}} (S \rightarrow \mathbf{N}).$$

Then

$$\forall h_1, h_2. (h_1 \in H) \wedge (h_2 \in H) \implies (h_1 \cup h_2 \in H).$$