

Homework 4: STLC — Solutions

Problem 1

(a)

$$M = \lambda f : \tau_1 \rightarrow \tau_2. \lambda x : \tau_1. \text{left}(f x)$$

Let

$$\Gamma = f : \tau_1 \rightarrow \tau_2, x : \tau_1$$

Then the typing derivation is:

$$\frac{\frac{\frac{f : \tau_1 \rightarrow \tau_2 \in \Gamma}{\Gamma \vdash f : \tau_1 \rightarrow \tau_2} \text{ (VAR)} \quad \frac{x : \tau_1 \in \Gamma}{\Gamma \vdash x : \tau_1} \text{ (VAR)}}{\Gamma \vdash f x : \tau_2} \text{ (APP)}}{\Gamma \vdash \text{left}(f x) : \tau_2 + \tau_3} \text{ (LEFT)} \quad \frac{}{f : \tau_1 \rightarrow \tau_2 \vdash \lambda x : \tau_1. \text{left}(f x) : \tau_1 \rightarrow (\tau_2 + \tau_3)} \text{ (ABS)} \quad \frac{}{\bullet \vdash M : (\tau_1 \rightarrow \tau_2) \rightarrow \tau_1 \rightarrow (\tau_2 + \tau_3)} \text{ (ABS)}$$

(b)

$$M = \lambda s : (\tau_1 \times \tau_2) + (\tau_3 \times \tau_4). \text{case } s \text{ do } (\lambda p : \tau_1 \times \tau_2. \text{left}(\text{proj1 } p)) \\ (\lambda q : \tau_3 \times \tau_4. \text{right}(\text{proj2 } q))$$

Let

$$\Gamma = s : (\tau_1 \times \tau_2) + (\tau_3 \times \tau_4)$$

Then the typing derivation is:

$$\frac{\frac{s : (\tau_1 \times \tau_2) + (\tau_3 \times \tau_4) \in \Gamma}{\Gamma \vdash s : (\tau_1 \times \tau_2) + (\tau_3 \times \tau_4)} \text{ (VAR)} \quad \mathcal{D}_{\text{left}} \quad \mathcal{D}_{\text{right}}}{\Gamma \vdash \text{case } s \text{ do } (\lambda p : \tau_1 \times \tau_2. \text{left}(\text{proj1 } p)) \\ (\lambda q : \tau_3 \times \tau_4. \text{right}(\text{proj2 } q)) : \tau_1 + \tau_4} \text{ (CASE)} \quad \frac{}{\bullet \vdash M : ((\tau_1 \times \tau_2) + (\tau_3 \times \tau_4)) \rightarrow (\tau_1 + \tau_4)} \text{ (ABS)}$$

Here $\mathcal{D}_{\text{left}}$ denotes the following derivation tree:

$$\frac{\frac{\frac{p : \tau_1 \times \tau_2 \in (\Gamma, p : \tau_1 \times \tau_2)}{\Gamma, p : \tau_1 \times \tau_2 \vdash p : \tau_1 \times \tau_2} \text{ (VAR)}}{\Gamma, p : \tau_1 \times \tau_2 \vdash \text{proj1 } p : \tau_1} \text{ (PROJ1)}}{\Gamma, p : \tau_1 \times \tau_2 \vdash \text{left}(\text{proj1 } p) : \tau_1 + \tau_4} \text{ (LEFT)} \quad \frac{}{\Gamma \vdash \lambda p : \tau_1 \times \tau_2. \text{left}(\text{proj1 } p) : (\tau_1 \times \tau_2) \rightarrow (\tau_1 + \tau_4)} \text{ (ABS)}$$

Use $\mathcal{D}_{\text{right}}$ to denote the following derivation tree:

$$\frac{\frac{\frac{q : \tau_3 \times \tau_4 \in (\Gamma, q : \tau_3 \times \tau_4)}{\Gamma, q : \tau_3 \times \tau_4 \vdash q : \tau_3 \times \tau_4} \text{ (VAR)}}{\Gamma, q : \tau_3 \times \tau_4 \vdash \text{proj2 } q : \tau_4} \text{ (PROJ2)}}{\Gamma, q : \tau_3 \times \tau_4 \vdash \text{right}(\text{proj2 } q) : \tau_1 + \tau_4} \text{ (RIGHT)} \quad \frac{}{\Gamma \vdash \lambda q : \tau_3 \times \tau_4. \text{right}(\text{proj2 } q) : (\tau_3 \times \tau_4) \rightarrow (\tau_1 + \tau_4)} \text{ (ABS)}$$

Problem 2

(a)

$$\frac{}{\Gamma \vdash \text{None} : \text{option } \tau} \text{ (T-NONE)} \quad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash \text{Some } M : \text{option } \tau} \text{ (T-SOME)}$$

$$\frac{\Gamma \vdash M : \text{option } \tau}{\Gamma \vdash \text{get } M : \tau} \text{ (T-GET)}$$

(b)

Variant	Preservation	Progress
1	holds	fails
2	holds	holds
3	holds	holds
4	fails	holds
5	holds	fails
6	fails	fails

Variant 1.

$$\bullet \vdash \text{Some}(\text{get None}) : \text{option T}$$

This term is not a value and cannot reduce without SOME.

Variant 4.

$$\bullet \vdash \text{get None} : \text{T} \quad \text{get None} \longrightarrow \text{None} \quad \bullet \not\vdash \text{None} : \text{T}$$

None has only option types.

Variant 5.

$$\bullet \vdash \text{get}(\text{get None}) : \text{T}$$

Here None has type option (option T). The inner get None is a value with no reduction. The whole term is neither a value nor able to reduce.

Variant 6.

$$\bullet \vdash \text{get}(\text{Some}(\lambda z : \text{T}. z)) : \text{option}(\text{T} \rightarrow \text{T})$$

$$\text{get}(\text{Some}(\lambda z : \text{T}. z)) \longrightarrow \lambda z : \text{T}. z$$

$$\bullet \not\vdash \lambda z : \text{T}. z : \text{option}(\text{T} \rightarrow \text{T})$$

The reduct has a function type, so preservation fails. Also,

$$\bullet \vdash \text{get}(\lambda z : \text{T}. z) : \text{T} \rightarrow \text{T}$$

This term is neither a value nor able to reduce, so progress fails.