Assignment on Lambda Calculus and Types

1. Consider the following two λ-expressions:

$$(\lambda f.\, \lambda x.\, f\,(f\,x))\,(\lambda b.\, \lambda x.\, \lambda y.\, b\,y\,x)\,(\lambda z.\, \lambda w.\, z)$$
$$(\lambda f.\, f\,f)\,(\lambda f.\, \lambda x.\, f\,(f\,x))$$

   For each of these expressions:

   (a) Give the normal order reduction sequence from this expression to its normal form.

   (b) Indicate the point where in the reduction sequence the first canonical form is reached.

   (c) Give the eager evaluation sequence.

   You can find the reduction rules in the lecture notes. Be careful to rename bound variables when necessary.

2. (Modified from the course exam in Autumn 2018)

   In this problem we add the option types to the simply-typed λ-calculus. We can use None and Some to construct terms of the option type, just like `None` and `Some` in Coq. Intuitively, None represents a dummy element (i.e. there is no meaningful element), $\mathsf{Some}\ M$ means that there is a meaningful element M, and $\mathsf{get}\ M$ gives us the meaningful element contained in M of the option type.

   Syntax:

$$\begin{array}{llll} (\mathsf{Types}) & \tau & ::= & \ldots \mid \mathsf{option}\ \tau \\ (\mathsf{Terms}) & M & ::= & \ldots \mid \mathsf{None} \mid \mathsf{Some}\ M \mid \mathsf{get}\ M \\ (\mathsf{Values}) & v & ::= & \ldots \mid \mathsf{None} \mid \mathsf{Some}\ v \end{array}$$

   Reduction rules:

$$\frac{M \to M'}{\mathsf{Some}\ M \to \mathsf{Some}\ M'}\ (\textsc{some}) \qquad \frac{M \to M'}{\mathsf{get}\ M \to \mathsf{get}\ M'}\ (\textsc{get-m})$$

$$\frac{}{\mathsf{get}\ (\mathsf{Some}\ M) \to M}\ (\textsc{get-some}) \qquad \frac{}{\mathsf{get}\ \mathsf{None} \to \mathsf{get}\ \mathsf{None}}\ (\textsc{get-none})$$

(a) Give 3 appropriate new typing rules, one for each new form of term. Note that your rules should ensure the preservation and progress theorems.

(b) Consider each of the following questions in isolation. Answer yes or no.

i. Suppose we remove the above (GET-M) rule.
Does the preservation theorem still hold?
Does the progress theorem still hold?

ii. Suppose we remove both the above (SOME) rule and the above (GET-M) rule.
Does the preservation theorem still hold?
Does the progress theorem still hold?

iii. Suppose we add the following rule.

$$\frac{}{\mathsf{get}\ v \to \mathsf{get}\ v}\ \text{(GET-V)}$$

Does the preservation theorem still hold?
Does the progress theorem still hold?

iv. Suppose we change the above (GET-SOME) rule to the following (GET-SOME') rule.

$$\frac{}{\mathsf{get}\ (\mathsf{Some}\ v) \to v}\ \text{(GET-SOME')}$$

Does the preservation theorem still hold?
Does the progress theorem still hold?

v. Suppose we change the above (GET-SOME) rule to the following (GET-SOME") rule.

$$\frac{}{\mathsf{get}\ (\mathsf{Some}\ v) \to \mathsf{Some}\ (\mathsf{get}\ v)}\ \text{(GET-SOME")}$$

Does the preservation theorem still hold?
Does the progress theorem still hold?

vi. Suppose we change the above (GET-NONE) rule to the following (GET-NONE') rule.

$$\frac{}{\mathsf{get}\ \mathsf{None} \to \mathsf{None}}\ \text{(GET-NONE')}$$

Does the preservation theorem still hold?
Does the progress theorem still hold?